THE STRUCTURE OF DOUBLE GROUPOIDS

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Abstract. We give a general description of the structure of a discrete double groupoid (with an extra, quite natural, filling condition) in terms of groupoid factorizations and groupoid 2-cocycles with coefficients in abelian group bundles. Our description goes as follows: To any double groupoid, we associate an abelian group bundle and a second double groupoid, its frame. The frame satisfies that every box is determined by its edges, and thus is called a 'slim' double groupoid. In a first step, we prove that every double groupoid is obtained as an extension of its associated abelian group bundle by its frame. In a second, independent, step we prove that every slim double groupoid with filling condition is completely determined by a factorization of a certain canonically defined 'diagonal' groupoid.

Introduction

The main result of this paper is the determination of the structure of a discrete double groupoid -satisfying a natural filling condition- in terms of groupoid data. By 'discrete' we mean here that no additional structure (differential, measurable, topological, etc.) is assumed. The problem of describing all double groupoids in terms of more familiar structures was explicitly raised by Brown and Mackenzie in [BM92, p. 271].

Double groupoids were introduced by Ehresmann [E63] in the early sixties, and later studied by several people because of their connection with different areas of mathematics, such as homotopy theory, differential geometry and Poisson-Lie groups. See for instance [B04, BJ04, BM92, BS76, L82, LW89, M92, M99, M00, P74, P77] and references therein.

A double groupoid is a groupoid object in the category of groupoids. This can be interpreted as a set of 'boxes' with two groupoid compositions –the *vertical* and *horizontal* compositions–, together with compatible groupoid

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compositions of the edges, obeying several conditions, in particular and most importantly the so called *interchange law*.

The double groupoids we are interested in satisfy the following:

Filling condition 0.1. For every configuration of matching edges as in Figure 1 (a), there is at least one box in the double groupoid as depicted in Figure 1 (b). This box is called the 'filling' of the given configuration.

Figure 1 (a) Figure 1 (b)

This condition is often assumed in the case of double groupoids arising in differential geometry, and is discussed by Mackenzie in [M00].

Concerning the structure of double groupoids, some very early results on 'special double groupoids with special connections' were obtained in [BS76]. For more general double groupoids, only those in a few classes were known to be determined by 'groupoid data'. One of them is the class of *vacant* double groupoids (*i.e.* those for which every configuration as in Figure 1 (a) has a unique filling): it was proved by Mackenzie in [M92, M00], that vacant double groupoids are essentially the same thing as *exact* factorizations of groupoids. Another one is the class of *transitive* or *locally trivial* double groupoids: in the paper [BM92], R. Brown and Mackenzie show that such a double groupoid is determined by its *core diagram*.

We now state our main result; unexplained notions and notations will be properly defined later in the text. Let $\begin{array}{ccc} \mathcal{B} & \rightrightarrows & \mathcal{H} \\ \downdownarrows & & \downdownarrows \\ \mathcal{V} & \rightrightarrows & \mathcal{P} \end{array}$ be a double groupoid. Let $\square(\mathcal{V},\mathcal{H})$ be the coarse double groupoid with sides in $\mathcal{V}$ and $\mathcal{H}$ and let $\Pi : \mathcal{B} \to \square(\mathcal{V},\mathcal{H})$ be the natural morphism of double groupoids. Let also

$$\begin{aligned} \mathbf{K} &:= \{K \in \mathcal{B} :\ t(K), b(K), l(K), r(K) \in \mathcal{P}\}, \\ \mathcal{F} &:= \text{image of } \Pi; \end{aligned}$$

$\mathcal{F}$ is called the frame of $\mathcal{B}$. The frame is what we call a *slim* double groupoid; that is, a double groupoid in which every box is uniquely determined by its edges. On the other hand, $\mathbf{K}$ is an abelian group bundle over $\mathcal{P}$ with respect to vertical composition, which coincides with the horizontal one in $\mathbf{K}$.

Theorem 1. *(a) The structure of the double groupoid $\mathcal{B}$ is determined completely by its associated abelian group bundle $\mathbf{K}$ and its frame $\mathcal{F}$, together with some extra cohomological data.*

(b) Assume that $\mathcal{B}$ is slim and satisfies the filling condition. Then there exists a groupoid $\mathcal{D}$ and morphisms of groupoids $i : \mathcal{H} \to \mathcal{D}$, $j : \mathcal{V} \to \mathcal{D}$ with

$\mathcal{D} = j(\mathcal{V})i(\mathcal{H})$ (unique up to isomorphisms), such that $\mathcal{B} \simeq \square(\mathcal{D}, j, i)$– as defined in Subsection 2.1. $\square$

Part (a) of Theorem 1 is contained in Theorem 1.9. Part (b) is contained in Theorem 2.8.

R. Brown has kindly pointed out to us that the construction of the double groupoid $\square(\mathcal{D}, j, i)$ was found by him a long time ago, and that it was taken up by Lu and Weinstein [LW89].

It is well-known that a discrete groupoid can be fully described in group-theoretical terms. Our main result shows that there is an analogous description, albeit more complicated, of the structure of discrete double groupoids in group-theoretical terms. Part (a) of Theorem 1, included to underline the completeness of our approach, does not require the filling condition and suggests a study of the corresponding cohomological theory, that we postpone to a future publication. Part (b) of Theorem 1 gives a description of a much larger class of double groupoids than the vacant double groupoids characterized in [M92, 2.15]. Namely, a vacant double groupoid is always slim, and if $\mathcal{B}$ is a slim double groupoid, then the following are equivalent:

(i) $\mathcal{B}$ is vacant.
(ii) $\mathcal{B}$ satisfies the filling condition, the morphisms i and j are injective, and the factorization is exact.

Thus, there are plenty of slim double groupoids satisfying the filling condition that are not vacant.

The extension of our main result to the context of Lie double groupoids, or other natural settings, requires some extra labor. The definition of the cocycles in part (a) requires the choice of a section, imposing an extra obstruction. The study of part (b) in the context of Lie double groupoids, under suitable natural hypotheses, has been recently carried out in [AOT], starting from our main result in Theorem 2.8.

In the paper [AN05] we proved that vacant finite double groupoids gave rise, in a natural way, to a class of tensor categories. Thus the results of [AN05] generalized a well-known construction in Hopf algebra theory studied, among others, by G. I. Kac, Majid and Takeuchi.

Later, in [AN06], this result was extended to the much more general class of finite double groupoids satisfying only the filling condition 0.1. It turns out that double groupoids giving rise through this construction to a special class of tensor categories called *fusion* categories must be slim. We discuss this class of double groupoids in more detail in the last section of the paper. We plan to apply the results of this last section to the determination of the corresponding fusion categories in a subsequent publication.

This paper is organized as follows. In Section 1, we recall the definition and special features of double groupoids. In Section 3 we discuss a special

class of slim double groupoids motivated by the paper [AN06]. In the Appendix we discuss an alternative approach to the corner functions introduced in [AN06].

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Notation. The cardinality of a set S will be indicated by $|S|$.

Throughout this paper we fix a nonempty set $\mathcal{P}$. A groupoid $\mathcal{G}$ on the base $\mathcal{P}$, with source s and target e, will be indicated by $s, e : \mathcal{G} \rightrightarrows \mathcal{P}$ or simply $\mathcal{G}$ if no confusion arises. Composition in $\mathcal{G}$ will be indicated by juxtaposition of arrows; so that if $g, h \in \mathcal{G}$, such that $e(g) = s(h)$, their composition will be denoted by $gh \in \mathcal{G}$. Let $P, Q \in \mathcal{P}$. As usual, $\mathcal{G}(P, Q)$ is the set of arrows from P to Q and $\mathcal{G}(Q) = \mathcal{G}(Q, Q)$. A map $p : \mathcal{E} \to \mathcal{P}$ will be called a fiber bundle. Let $p : \mathcal{E} \to \mathcal{P}$ be a fiber bundle. Recall that a *left action* of $\mathcal{G}$ on p is a map $\triangleright : \mathcal{G} \,{}_e\!\times_p \mathcal{E} \to \mathcal{E}$ such that

$$p(g \triangleright x) = s(g), \qquad g \triangleright (h \triangleright x) = gh \triangleright x, \qquad \mathbf{id}\, p(x) \triangleright x = x,$$

for all $g, h \in \mathcal{G}$, $x \in \mathcal{E}$ composable in the appropriate sense. Hence, if $\mathcal{E}_b := p^{-1}(b)$, then the action of $g \in \mathcal{G}$ is an isomorphism $g \triangleright \underline{\quad} : \mathcal{E}_{t(g)} \to \mathcal{E}_{s(g)}$. Let $x \in \mathcal{E}$ and define

$$\begin{aligned} \mathcal{O}_x &= \{g \triangleright x : g \in \mathcal{G}, e(g) = p(x)\}, && \text{the orbit of } x, \\ \mathcal{G}^x &= \{g \in \mathcal{G} : g \triangleright x = x\} < \mathcal{G}(p(x)), && \text{the isotropy subgroup of } x. \end{aligned}$$

1. Double groupoids

Let $\mathcal{B}$ be a *double groupoid* [E63]; we follow the conventions and notations from [AN05, Section 2] and [AN06, Section 1]. As usual, we represent $\mathcal{B}$ in the form of four related groupoids

$$\begin{matrix} \mathcal{B} & \rightrightarrows & \mathcal{H} \\ \downdownarrows & & \downdownarrows \\ \mathcal{V} & \rightrightarrows & \mathcal{P} \end{matrix}$$

subject to a set of axioms. The source and target maps of these groupoids are indicated by $t, b : \mathcal{B} \to \mathcal{H}$; $r, l : \mathcal{B} \to \mathcal{V}$; $r, l : \mathcal{H} \to \mathcal{P}$; $t, b : \mathcal{V} \to \mathcal{P}$

('top', 'bottom', 'right' and 'left'). An element $A \in \mathcal{B}$ is depicted as a box

$$A = l \begin{array}{c} t \\ \boxed{} \\ b \end{array} r$$

where $t(A) = t$, $b(A) = b$, $r(A) = r$, $l(A) = l$, and the four vertices of the square representing A are $tl(A) = lt(A)$, $tr(A) = rt(A)$, $bl(A) = lb(A)$, $br(A) = rb(A)$. The notation $A|B$ means that $r(A) = l(B)$ (A and B are horizontally composable); the corresponding horizontal product is denoted AB. Similarly, $\dfrac{A}{B}$ means that $b(A) = t(B)$ (A and B are vertically composable) and the vertical product is denoted $\begin{matrix} A \\ B \end{matrix}$.

The notation $A = \begin{array}{|c|}\hline\hline \\ \hline\end{array}$ means that $t(A)$ is an identity; analogously, $B = \begin{array}{||c|}\hline \\ \hline\end{array}$ means that $l(B)$ is an identity, etc.

These four groupoids should satisfy certain axioms, see e. g. [AN05]. In particular, $\mathbf{id}\,\mathrm{id}_{\mathcal{H}} P = \mathbf{id}\,\mathrm{id}_{\mathcal{V}} P$, for any $P \in \mathcal{P}$; this box is denoted Θ_P and clearly it is of the form $\begin{array}{||c||}\hline\hline \\ \hline\hline\end{array}$.

1.1. **Core groupoids.**

We recall the core groupoid $\mathbf{E}$ of $\mathcal{B}$ introduced by Brown and Mackenzie [M00, BM92]. See also [AN06]. Let

$$\mathbf{E} := \{E \in \mathcal{B} :\ r(E), t(E) \in \mathcal{P}\}.$$

Thus, elements of $\mathbf{E}$ are of the form $\begin{array}{|c||}\hline\hline \\ \hline\end{array}$. There is a groupoid structure $s, e : \mathbf{E} \rightrightarrows \mathcal{P}$, $s(E) = bl(E)$, $e(E) = br(E)$, $E \in \mathbf{E}$, identity map $\mathrm{id} : \mathcal{P} \to \mathbf{E}$, $P \mapsto \Theta_P$, and composition $\mathbf{E}\,{}_e\!\times_s \mathbf{E} \to \mathbf{E}$, given by

$$E \circ M := \left\{\begin{matrix} \mathbf{id}l(M) & M \\ E & \mathbf{id}b(M) \end{matrix}\right\}, \tag{1.1}$$

$M, E \in \mathbf{E}$. The inverse of $E \in \mathbf{E}$ is $E^{(-1)} := (E\mathbf{id}b(E)^{-1})^v = \left\{\begin{matrix} \mathbf{id}l(E)^{-1} \\ E^h \end{matrix}\right\}$.

If $Q \in \mathcal{P}$, the group $\mathbf{E}(Q)$ consists of boxes in $\mathcal{B}$ of the form $h\begin{array}{|c||}\hline\hline \\ \hline\end{array}\!\!\!\!\raisebox{-1.2em}{y}$, with $y \in \mathcal{H}(Q)$, $h \in \mathcal{V}(Q)$.

If $(x, g) \in \mathcal{H}\,{}_r\!\times_t \mathcal{V}$ and $B \in \mathcal{B}$ then we consider the sets of boxes with fixed 'upper and right' sides

$$\mathcal{UR}(x, g) = \left\{U \in \mathcal{B} :\ U = \begin{array}{c} x \\ \boxed{} \end{array} g \right\}, \quad \mathcal{UR}(B) = \mathcal{UR}(t(B), r(B)). \tag{1.2}$$

Let $\gamma : \mathcal{B} \to \mathcal{P}$ be the 'left-bottom' vertex, $\gamma(B) = lb(B)$.

Proposition 1.1. *(a). There is an action of* $\mathbf{E}$ *on* $\gamma : \mathcal{B} \to \mathcal{P}$ *given by*

$$E \rightharpoondown A := \begin{Bmatrix} \mathbf{id} l(A) & A \\ E & \mathbf{id} b(A) \end{Bmatrix}, \quad A \in \mathcal{B}, E \in \mathbf{E}. \tag{1.3}$$

(b). Let $B \in \mathcal{B}$*. Then* $\mathcal{O}_B = \mathcal{UR}(B)$ *and* $\mathbf{E}^B$ *is trivial.*

Proof. (a) is straightforward. (b): it follows from Definition (1.3) that $\mathcal{O}_B \subseteq \mathcal{UR}(B)$. Then observe that for any $C \in \mathcal{UR}(B)$, there exists a unique $E \in \mathbf{E}$ such that $E \rightharpoondown B = C$, namely $E = \begin{Bmatrix} B^v \\ C \end{Bmatrix} \operatorname{id} b(B)^{-1}$. □

1.2. **The associated abelian group bundle.**

An important invariant of a double groupoid is the intersection $\mathbf{K}$ of all four core groupoids:

$$\mathbf{K} := \{K \in \mathcal{B} : \ t(K), b(K), l(K), r(K) \in \mathcal{P}\}.$$

Thus a box is in $\mathbf{K}$ if and only if it is of the form . Let $p : \mathbf{K} \to \mathcal{P}$ be the 'common vertex' function, say $p(K) = lb(K)$. For any $P \in \mathbf{K}$, let $\mathbf{K}(P)$ be the fiber at P; $\mathbf{K}(P)$ is an abelian group under vertical composition, that coincides with horizontal composition. This is just the well-known fact: "a double group is the same as an abelian group". Indeed, apply the interchange law

$$\begin{matrix} (KL) \\ (MN) \end{matrix} = \begin{pmatrix} K \\ M \end{pmatrix} \begin{pmatrix} L \\ N \end{pmatrix}$$

to four boxes $K, L, M, N \in \mathbf{K}(P)$: if $L = M = \Theta_P$, this says that $\begin{matrix} K \\ N \end{matrix} = KN$ and the two operations coincide. If, instead, $K = N = \Theta_P$, this says that $\begin{matrix} L \\ M \end{matrix} = ML$, hence the composition is abelian. Note that this operation in $\mathbf{K}$ coincides also with the core multiplication (1.1). In short, $\mathbf{K}$ is an abelian group bundle over $\mathcal{P}$.

The vertical and the horizontal groupoids $\mathcal{V}$ and $\mathcal{H}$ act on $\mathbf{K}$ by vertical, respectively horizontal, conjugation:

(1.4) If $g \in \mathcal{V}(Q, P)$ and $K \in \mathbf{K}(P)$ then $g \cdot K := \begin{matrix} \operatorname{id} g \\ K \\ \operatorname{id} g^{-1} \end{matrix} \in \mathbf{K}(Q)$.

(1.5) If $x \in \mathcal{H}(Q, P)$ and $K \in \mathbf{K}(P)$ then $x \cdot K = \operatorname{id} x K \operatorname{id} x^{-1} \in \mathbf{K}(Q)$.

Both actions are by group bundle automorphisms.

1.3. **Frame of a double groupoid.**

The frame of a double groupoid, as said in the Introduction, is the image of the natural map into the coarse double groupoid of the side groupoids. In this subsection, we made this statement precise; then we show that the initial double groupoid is determined by its frame, its associated abelian

group bundle and some actions and cocycles. In the next subsection we explain this fact in terms of extensions.

Let $\mathcal{P}$ be a set and $\mathcal{V}$, $\mathcal{H}$ be groupoids over $\mathcal{P}$ denoted vertically and horizontally, respectively. Let $\square(\mathcal{V},\mathcal{H})$ be the set of quadruples $\begin{pmatrix} & x & \\ f & & g \\ & y & \end{pmatrix}$ with $x, y \in \mathcal{H}$, $f, g \in \mathcal{V}$ such that

$$l(x) = t(f), \quad r(x) = t(g), \quad l(y) = b(f), \quad r(y) = b(g).$$

If no confusion arises, we shall denote a quadruple as above by a box $h \overset{x}{\underset{y}{\square}} g$.

The collection $\begin{array}{ccc} \square(\mathcal{V},\mathcal{H}) & \rightrightarrows & \mathcal{H} \\ \downdownarrows & & \downdownarrows \\ \mathcal{V} & \rightrightarrows & \mathcal{P} \end{array}$ forms a double groupoid, called the *coarse double groupoid* with sides in $\mathcal{V}$ and $\mathcal{H}$, with horizontal and vertical compositions given by

$$h \overset{x}{\underset{y}{\square}} g \, g \overset{x'}{\underset{y'}{\square}} g' = h \overset{xx'}{\underset{yy'}{\square}} g', \qquad \begin{array}{c} h \overset{x}{\underset{y}{\square}} g \\ h' \overset{y}{\underset{y'}{\square}} g' \end{array} = hh' \overset{x}{\underset{y'}{\square}} gg',$$

for all $x, y, x', y' \in \mathcal{H}$, $g, h, g', h' \in \mathcal{V}$ appropriately composable.

Let $\begin{array}{ccc} \mathcal{B} & \rightrightarrows & \mathcal{H} \\ \downdownarrows & & \downdownarrows \\ \mathcal{V} & \rightrightarrows & \mathcal{P} \end{array}$ be a double groupoid. There is a map $\Pi : \mathcal{B} \to \square(\mathcal{V},\mathcal{H})$ given by

$$\Pi\left(f \overset{x}{\underset{y}{\square}} g \right) = \begin{pmatrix} & x & \\ f & & g \\ & y & \end{pmatrix}, \qquad f \overset{x}{\underset{y}{\square}} g \in \mathcal{B}.$$

Clearly, Π induces a morphism of double groupoids $\mathcal{B} \to \square(\mathcal{V},\mathcal{H})$.

Definition 1.2. We shall say that $\mathcal{B}$ is *slim* if Π is injective (any box is determined by its four sides).

Let $\mathcal{F}$ be the image of Π. The *frame* of $\mathcal{B}$ is the slim double groupoid

$$\begin{array}{ccc} \mathcal{F} & \rightrightarrows & \mathcal{H} \\ \downdownarrows & & \downdownarrows. \\ \mathcal{V} & \rightrightarrows & \mathcal{P} \end{array}$$

Let $\mathcal{B}$ be a double groupoid. Several properties of $\mathcal{B}$ are controlled by its frame $\mathcal{F}$. Recall the following definitions [BM92, Definition 2.3].

(a) $\mathcal{B}$ is *horizontally transitive* if every configuration of matching sides $\sqcup$ can be completed to at least one box in $\mathcal{B}$.

(b) $\mathcal{B}$ is *vertically transitive* if every configuration of matching sides can be completed to at least one box in $\mathcal{B}$.
(c) $\mathcal{B}$ is *transitive* or *locally trivial* if it is both vertically and horizontally transitive.

Remark 1.3. Let $\mathcal{B}$ be a double groupoid. Then

(i) $\mathcal{B}$ satisfies the filling condition 0.1 if and only if $\mathcal{F}$ does so.
(ii) $\mathcal{B}$ is horizontally (vertically) transitive if and only if so $\mathcal{F}$ is so.
(iii) If $\mathcal{B}$ is vacant then it is slim.

We are now ready to prove our first basic result. *Let us fix a section* $\mu : \mathcal{F} \to \mathcal{B}$ *of* Π. Recall the action $\rightharpoondown$ of the core groupoid $\mathbf{E}$ on $\gamma : \mathcal{B} \to \mathcal{P}$ defined in (1.3).

Lemma 1.4. *The map* $\Psi : \mathbf{K}\,{}_p\!\times_\gamma \mathcal{F} \to \mathcal{B}$ *given by* $\Psi(K, F) = K \rightharpoondown \mu(F)$ *is a bijection.*

Proof. In other words, we have to show that for any $B \in \mathcal{UR}(\mu(F))$, there is a unique $K \in \mathbf{K}$ such that $B = K \rightharpoondown \mu(F)$. Note that $\Pi(K \rightharpoondown \mu(F)) = F$. Hence, $K \rightharpoondown \mu(F) = K' \rightharpoondown \mu(F')$ implies $F = F'$; thus $K \rightharpoondown \mu(F) = K' \rightharpoondown \mu(F)$, and $K = K'$ by Proposition 1.1 (b). That is, Ψ is injective. We show that it is surjective. Let $B \in \mathcal{B}$ and let $F = \Pi(B)$. Since B and $\mu(F)$ have the same sides, there exists $K \in \mathbf{E}$ such that $B = K \rightharpoondown \mu(F)$, again by Proposition 1.1. But clearly $K \in \mathbf{K}$. □

We next introduce vertical and horizontal cocycles to control the lack of multiplicativity of the section μ. We define $\tau : \mathcal{F}\,{}_r\!\times_l \mathcal{F} \to \mathbf{K}$ and $\sigma : \mathcal{F}\,{}_b\!\times_t \mathcal{F} \to \mathbf{K}$ by

$$\mu(F)\mu(G) = \tau(F, G) \rightharpoondown \mu(FG), \qquad r(F) = l(G), \tag{1.6}$$

$$\begin{matrix}\mu(F)\\ \mu(G)\end{matrix} = \sigma(F, G) \rightharpoondown \mu\begin{pmatrix}F\\ G\end{pmatrix}, \qquad b(F) = t(G). \tag{1.7}$$

That is,

$$\mu(F)\mu(G) = \left\{\begin{matrix}\operatorname{id} l(F) & \mu(FG)\\ \tau(F,G) & \operatorname{id} b(FG)\end{matrix}\right\}, \qquad \begin{matrix}\mu(F)\\ \mu(G)\end{matrix} = \left\{\begin{matrix}\operatorname{id} l(F)l(G) & \mu\begin{pmatrix}F\\ G\end{pmatrix}\\ \sigma(F,G) & \operatorname{id} b(G)\end{matrix}\right\},$$

for appropriate F, $G \in \mathcal{F}$. The cocycles σ and τ are well-defined in virtue of Lemma 1.4. If we assume that $\mu(\operatorname{id} x) = \operatorname{id} x$ and $\mu(\operatorname{id} g) = \operatorname{id} g$ for any $x \in \mathcal{H}$ and $g \in \mathcal{V}$ then σ and τ are normalized:

$$\tau(F, \operatorname{id} r(F)) = \Theta_{bl(F)} = \tau(\operatorname{id} l(F), F), \tag{1.8}$$

$$\sigma(F, \operatorname{id} b(F)) = \Theta_{bl(F)} = \sigma(\operatorname{id} t(F), F). \tag{1.9}$$

Now we can reconstruct the horizontal and vertical products of $\mathcal{B}$ in terms of the associated abelian group bundle $\mathbf{K}$, the frame slim double groupoid

$\mathcal{F}$, the actions (1.5), (1.4) and the cocycles σ and τ. If $K, L \in \mathbf{K}$, $F, G \in \mathcal{F}$ then

$$\big(K \rightharpoondown \mu(F)\big)\big(L \rightharpoondown \mu(G)\big) = \big(K(b(F)\cdot L)\tau(F,G)\big) \rightharpoondown \mu(FG), \tag{1.10}$$

if $r(F) = l(G)$ and

$$\begin{Bmatrix} K \rightharpoondown \mu(F) \\ L \rightharpoondown \mu(G) \end{Bmatrix} = \big((l(g)^{-1}\cdot K)\, L\, \sigma(F,G)\big) \rightharpoondown \mu \begin{pmatrix} F \\ G \end{pmatrix} \tag{1.11}$$

if $b(F) = t(G)$.

1.4. **Extensions of double groupoids by abelian group bundles.**

Our aim now is to interpret Lemma 1.4 and formulas (1.10), (1.11) in cohomological terms. The description in the preceding subsection suggests the following construction.

Let $\begin{matrix} \mathcal{F} & \rightrightarrows & \mathcal{H} \\ \downdownarrows & & \downdownarrows \\ \mathcal{V} & \rightrightarrows & \mathcal{P} \end{matrix}$ be any double groupoid (not necessarily slim) and let $\gamma : \mathbf{K} \to \mathcal{P}$ be any abelian group bundle. Assume that $\mathcal{V}$ and $\mathcal{H}$ act on $\mathbf{K}$ by group bundle isomorphisms. Let $\tau : \mathcal{F}\,{}_r\!\times_l \mathcal{F} \to \mathbf{K}$ and $\sigma : \mathcal{F}\,{}_b\!\times_t \mathcal{F} \to \mathbf{K}$ be maps such that

$$\gamma(\sigma(F,G)) = bl(G), \qquad \text{if } b(F) = t(G), \tag{1.12}$$

$$\gamma(\tau(F,G)) = bl(F), \qquad \text{if } r(F) = l(G), \tag{1.13}$$

normalized by (1.8) and (1.9). Consider the collection $\begin{matrix} \mathbf{K}\,{}_p\!\times_\gamma \mathcal{F} & \rightrightarrows & \mathcal{H} \\ \downdownarrows & & \downdownarrows \\ \mathcal{V} & \rightrightarrows & \mathcal{P} \end{matrix}$, where:

- The maps t, b, l, r on $\mathbf{K}\,{}_p\!\times_\gamma \mathcal{F}$ are defined by those in $\mathcal{F}$: $t(K, F) = t(F)$ and so on.
- The horizontal and vertical products in $\mathbf{K}\,{}_p\!\times_\gamma \mathcal{F}$ are given by

$$(K,F)(L,G) = \big(K\,(b(F)\cdot L)\,\tau(F,G), FG\big), \qquad \text{if } F|G, \tag{1.14}$$

$$\begin{matrix} (K,F) \\ (L,G) \end{matrix} = \left((l(G)^{-1}\cdot K)\, L\, \sigma(F,G), \begin{matrix} F \\ G \end{matrix}\right), \qquad \text{if } \frac{F}{G}. \tag{1.15}$$

- The identity maps $\mathrm{id} : \mathcal{V} \to \mathbf{K}\,{}_p\!\times_\gamma \mathcal{F}$, $\mathrm{id} : \mathcal{H} \to \mathbf{K}\,{}_p\!\times_\gamma \mathcal{F}$ are given by $\mathrm{id}\, g = (\Theta_{b(g)}, \mathrm{id}\, g)$, $\mathrm{id}\, x = (\Theta_{l(x)}, \mathrm{id}\, x)$, $g \in \mathcal{V}$, $x \in \mathcal{H}$.
- The inverse of (K, F) with respect to the horizontal and vertical products are respectively given by

$$(K,F)^h = \left(b(F)^{-1}\cdot \left(K^{-1}\tau(F,F^h)^{-1}\right), F^h\right), \tag{1.16}$$

$$(K,F)^v = \left((l(F)\cdot K)^{-1}\,\sigma(F,F^v)^{-1}, F^v\right). \tag{1.17}$$

Proposition 1.5. *$\mathbf{K}\,{}_p\!\times_\gamma \mathcal{F}$ is a double groupoid if and only if, for all $F, G, H \in \mathcal{F}$,*

$$\tau(F,G)\,\tau(FG,H) = \tau(F,GH)\,\big(b(F)\cdot\tau(G,H)\big), \quad F|G|H; \tag{1.18}$$

$$\sigma(G,H)\,\sigma\begin{pmatrix} F, \begin{matrix} G \\ H \end{matrix} \end{pmatrix} = \big(l(H)^{-1}\cdot\sigma(F,G)\big)\,\sigma\begin{pmatrix} \begin{matrix} F \\ G \end{matrix}, H \end{pmatrix}, \quad \begin{array}{c} F \\ \hline G \\ \hline H \end{array}; \tag{1.19}$$

$$l(H)^{-1}\cdot(t(H)\cdot L) = b(H)\cdot(r(H)^{-1}\cdot L), \quad L \in \mathbf{K}(tr(H)); \tag{1.20}$$

$$\begin{aligned} &\big(l(H)^{-1}\cdot\tau(F,G)\big)\tau(H,J)\sigma(FG,HJ) \\ &\qquad = \big(b(H)\cdot\sigma(G,J)\big)\sigma(F,H)\tau\begin{pmatrix} F & G \\ H, & J \end{pmatrix}, \quad \begin{array}{c|c} F & G \\ \hline H & J \end{array}. \end{aligned} \tag{1.21}$$

Definition 1.6. If the conditions in Proposition 1.5 hold, we say that the double groupoid $\mathbf{K}\,{}_p\!\times_\gamma \mathcal{F}$ is an *abelian extension* of the abelian group bundle $\mathbf{K}$ by $\mathcal{F}$.

Proof. The associativity of the horizontal and vertical compositions are respectively equivalent to (1.18) and (1.19).

We have to check the axioms of double groupoid as in [BS76]; we follow [AN05, Lemma 1.2]. All the axioms are consequences of the definitions (since the axioms hold in $\mathcal{F}$) except the interchange law, which is equivalent to (1.20) and (1.21). Indeed, let $H \in \mathcal{F}$ and $L \in \mathbf{K}(tr(H))$. Computing $\begin{Bmatrix} \operatorname{id} t(H) & L \\ H & \operatorname{id} r(H) \end{Bmatrix}$ in two different ways, we see that (1.20) is equivalent to the interchange law in this case. Next, consider

$$(K,F),\ (L,G),\ (M,H),\ (N,J) \in \mathbf{K}\,{}_p\!\times_\gamma \mathcal{F} \text{ such that } \begin{array}{c|c} F & G \\ \hline H & J \end{array}.$$

Compute $\begin{matrix} \{(K,F)(L,G)\} \\ \{(M,H)(N,J)\} \end{matrix}$ and $\begin{Bmatrix} (K,F) \\ (M,H) \end{Bmatrix}\begin{Bmatrix} (L,G) \\ (N,J) \end{Bmatrix}$. The resulting expressions are equal if and only if

$$\begin{aligned} &l(H)^{-1}\cdot(b(F)\cdot L)l(H)^{-1}\cdot\tau(F,G)\tau(H,J)\sigma(FG,HJ) \\ &\qquad = b(H)\cdot(l(J)^{-1}\cdot L)b(H)\cdot\sigma(G,J)\sigma(F,H)\tau\begin{pmatrix} F & G \\ H, & J \end{pmatrix} \end{aligned}$$

It is not difficult to see that this is equivalent to (1.20) and (1.21). $\square$

Remark 1.7. As we shall see later, see Section 2, the slim double groupoid $\mathcal{F}$ determines a 'diagonal' groupoid $\mathcal{D}$ endowed with groupoid maps $j : \mathcal{V} \to \mathcal{D}$,

$i : \mathcal{H} \to \mathcal{D}$. In terms of this groupoid, Condition (1.20) means that the actions of $\mathcal{H}$ and $\mathcal{V}$ come from an action of $\mathcal{D}$ on $\mathbf{K}$.

Remark 1.8. Conditions (1.18) and (1.19) are cocycle conditions on the horizontal and vertical composition groupoids. Together with (1.21) they give a cocycle condition in the double complex associated to the double groupoid $\mathcal{F}$ as considered in [AN05, AM06].

Assume that the hypotheses in Proposition 1.5 are fulfilled. Let us identify $\mathbf{K}$ with a subset of $\mathbf{K}\,{}_p\!\times_\gamma \mathcal{F}$ via $K \mapsto (K, \Theta_P)$, if $K \in \mathbf{K}(P)$. Also, let $\mu : \mathcal{F} \to \mathbf{K}\,{}_p\!\times_\gamma \mathcal{F}$, $\mu(F) = (\Theta_{bl(F)}, F)$. Then

$$(K, F) = \begin{matrix} \operatorname{id} l(F) & \mu(F) \\ K & \operatorname{id} b(F) \end{matrix}, \qquad \text{for any } (K, F) \in \mathbf{K}\,{}_p\!\times_\gamma \mathcal{F}.$$

Hence the formulas (1.14) and (1.15) are equivalent to (1.10) and (1.11), respectively. In particular we have

Theorem 1.9. *Any double groupoid is an abelian extension of its associated abelian group bundle by its frame.* □

2. Slim double groupoids and factorizations of groupoids

Let $\mathcal{V}$ and $\mathcal{H}$ be groupoids over $\mathcal{P}$. In this section we describe all slim double groupoids satisfying the filling condition 0.1 whose groupoids of vertical and horizontal edges coincide with $\mathcal{V}$ and $\mathcal{H}$, respectively.

2.1. **Double groupoid associated to a diagram of groupoids.**

Let us say that a *diagram* $(\mathcal{D}, j, i)$ over $\mathcal{V}$ and $\mathcal{H}$ is a groupoid $\mathcal{D}$ over $\mathcal{P}$ endowed with groupoid maps over $\mathcal{P}$

$$i : \mathcal{H} \to \mathcal{D}, \quad j : \mathcal{V} \to \mathcal{D}.$$

The class of all diagrams $(\mathcal{D}, j, i)$ over $\mathcal{V}$ and $\mathcal{H}$ is a category with morphisms $(\mathcal{D}, j, i) \to (\mathcal{D}', j', i')$ being morphisms $f : \mathcal{D} \to \mathcal{D}'$ of groupoids over $\mathcal{P}$ such that $fi = i'$ and $fj = j'$.

Consider the full subcategory of diagrams $(\mathcal{D}, j, i)$ with $\mathcal{D} = j(\mathcal{V})i(\mathcal{H})$; that is, such that every arrow in $\mathcal{D}$ can be written as a product $j(g)i(x)$, for some $g \in \mathcal{V}$, $x \in \mathcal{H}$, with $b(g) = l(x)$. An object in this subcategory will be called a $(\mathcal{V}, \mathcal{H})$-*factorization* of $\mathcal{D}$.

Each diagram $(\mathcal{D}, j, i)$ has an associated double groupoid $\square(\mathcal{D}, j, i)$ defined as follows. Boxes in $\square(\mathcal{D}, j, i)$ are of the form

$$A = h \underset{y}{\overset{x}{\square}} g \in \square(\mathcal{V}, \mathcal{H}),$$

with $x, y \in \mathcal{H}$, $g, h \in \mathcal{V}$, such that

$$i(x)j(g) = j(h)i(y) \quad \text{ in } \mathcal{D}.$$

Notice that $\square(\mathcal{D}, j, i)$ is stable under vertical and horizontal products in $\square(\mathcal{V}, \mathcal{H})$; therefore it is itself a double groupoid. By its very definition, $\square(\mathcal{D}, j, i)$ is *slim*.

The assignment $(\mathcal{D}, j, i) \to \square(\mathcal{D}, j, i)$ just defined is clearly functorial.

Example 2.1. Let G be a simply connected Poisson-Lie group, $\mathfrak{g}$ its Lie algebra. It is well known that $\mathfrak{g}$ is a Lie bialgebra; let $\mathfrak{g}^*$ be the dual Lie algebra and let $\mathfrak{d}$ be the correspondig Drinfeld double. Let G^* and D be simply connected Lie groups with Lie algebras $\mathfrak{g}^*$ and $\mathfrak{d}$, respectively. Then the maps $G \to D$ and $G^* \to D$ give rise to a double symplectic groupoid [LW89, Theorem 3].

Lemma 2.2. *The core groupoid of* $\square(\mathcal{D}, j, i)$ *is isomorphic to the groupoid* $\mathcal{V}^{\mathrm{op}}{}_j \times_i \mathcal{H} := \{(g, x) \in \mathcal{V}^{\mathrm{op}}{}_b \times_l \mathcal{H} : \, j(g) = i(x^{-1})\} \subseteq \mathcal{V}^{\mathrm{op}}{}_b \times_l \mathcal{H}$.

Proof. An isomorphism is given by the map $\mathbf{E} \to \mathcal{V}^{\mathrm{op}}{}_j \times_i \mathcal{H}$, defined by $E \mapsto (l(E), b(E))$. This map is surjective by construction of $\square(\mathcal{D}, j, i)$; it is injective as a consequence of the slim condition on $\square(\mathcal{D}, j, i)$. $\square$

Remark 2.3. If $\mathcal{D} = j(\mathcal{V})i(\mathcal{H})$ is a factorization, then $\square(\mathcal{D}, j, i)$ satisfies the filling condition 0.1. Indeed, if $g \in \mathcal{V}$, $x \in \mathcal{H}$, are such that $r(x) = t(g)$, then the condition $\mathcal{D} = j(\mathcal{V})i(\mathcal{H})$ implies that there exist $y \in \mathcal{H}$, $h \in \mathcal{V}$, such that $j(h)i(y) = i(x)j(g)$. Then, by construction, the box $h \begin{array}{c} x \\ \boxed{} \\ y \end{array} g$ is a filling in $\square(\mathcal{D}, j, i)$ for $\begin{array}{c} x \\ \rceil \end{array} g$.

Example 2.4. Suppose $\mathcal{D} = \mathcal{P}^2$ is the coarse groupoid on $\mathcal{P}$. Let the maps $i : \mathcal{H} \to \mathcal{D}$, $j : \mathcal{V} \to \mathcal{D}$, be defined by $i(x) = (l(x), r(x))$, $x \in \mathcal{H}$, and $j(g) = (t(g), b(g))$, $g \in \mathcal{V}$. Let $\mathcal{B} = \square(\mathcal{P}^2, j, i)$ be the associated double groupoid. The relations $i(x)j(g) = j(h)i(y)$ are satisfied in $\mathcal{D}$, for all $x, y \in \mathcal{H}$, $g, h \in \mathcal{V}$, such that $r(x) = t(g)$, $b(h) = l(y)$, $l(x) = t(h)$, $r(y) = b(g)$. Hence, for all such x, y, g, h there is a box $h \begin{array}{c} x \\ \boxed{} \\ y \end{array} g$ in $\mathcal{B}$. According to the composition rules in $\mathcal{B}$, it turns out that $\mathcal{B}$ is exactly the *coarse* double groupoid $\square(\mathcal{V}, \mathcal{H})$.

We shall show that double groupoids of the form $\square(\mathcal{D}, j, i)$ exhaust the class of slim double groupoids which satisfy the filling condition 0.1.

2.2. **Free product of groupoids.**

We now need to recall the basic properties of the free product construction for groupoids. We refer the reader to [H71] for a detailed exposition; this will be used in the definition of the diagonal groupoid in Subsection 2.3 and in the proof of Theorem 2.4.

In this subsection we work in the category of groupoids over $\mathcal{P}$; morphisms in this category are the identity on $\mathcal{P}$. Subgroupoids with the same base $\mathcal{P}$ are often called 'wide'.

Let $\mathcal{V}$ and $\mathcal{H}$ be groupoids, alluded to as 'vertical' and 'horizontal', respectively. Let $\mathcal{V} = \langle X|R\rangle$, $\mathcal{H} = \langle Y|S\rangle$, be presentations of $\mathcal{V}$ and $\mathcal{H}$ by generators and relations [H71, Chapter 9]. Let $\mathcal{V} * \mathcal{H} = \langle X \coprod Y | R \coprod S\rangle$ be the *free product* of the groupoids $\mathcal{V}$ and $\mathcal{H}$; $\mathcal{V} * \mathcal{H}$ is the coproduct of $\mathcal{V}$ and $\mathcal{H}$ in the category of groupoids over $\mathcal{P}$. In other words, the groupoid $\mathcal{V} * \mathcal{H}$ is characterized by the following universal property: for every groupoid $\mathcal{G}$ and groupoid maps $i : \mathcal{H} \to \mathcal{G}$, $j : \mathcal{V} \to \mathcal{G}$, there is a unique morphism of groupoids $f : \mathcal{V} * \mathcal{H} \to \mathcal{G}$ such that $f|_{\mathcal{V}} = j$, and $f|_{\mathcal{H}} = j$. In particular it does not depend on the choice of the presentations of $\mathcal{V}$ and $\mathcal{H}$.

Note that our free product of $\mathcal{V}$ and $\mathcal{H}$ is close to, but not the same as, the free product in [H71, Chapter 9]; precisely, it is the free product with amalgamation of identities from *loc. cit.*

An alternative way of describing the free product is the following. Consider the set $\operatorname{Path}(\mathcal{Q})$ of all paths of the quiver $\mathcal{Q} = \mathcal{H} \coprod \mathcal{V}$. An element in $\operatorname{Path}(\mathcal{Q})$ is either a an element $P \in \mathcal{P}$ that will be indicated by $[P]$, or a sequence $(u_1, \dots, u_n)$, $n \geq 1$, with $u_i \in \mathcal{Q}$, $e(u_i) = s(u_{i+1})$.

A path $U \in \operatorname{Path}(\mathcal{Q})$ is called *reduced* if either $U = [P]$, $P \in \mathcal{P}$, or $U = (u_1, \dots, u_n)$, $n \geq 1$, $u_i \in \mathcal{Q}$, and the following conditions hold:

- no u_i is an identity arrow,
- u_i and u_{i+1} do not belong to the same groupoid $\mathcal{H}$ or $\mathcal{V}$.

For instance, the horizontal identity $\operatorname{id}_{\mathcal{H}} P \in \mathcal{H}$, $P \in \mathcal{P}$, is a path which is not reduced.

Every path $(u_1, u_2, \dots, u_n)$, with $n > 0$, can be transformed into a reduced path by means of a finite number of reductions, that is, operations of one of the following types:

- removing u_i if u_i is an identity arrow and $n > 1$,
- replacing (u_1) by $[P]$ if $u_1 = \operatorname{id}_{\mathcal{H}} P$ or $\operatorname{id}_{\mathcal{V}} P$, $P \in \mathcal{P}$,
- replacing $(u_1, \dots, u_i, u_{i+1}, \dots, u_n)$ by $(u_1, \dots, u_i u_{i+1}, \dots, u_n)$, if u_i and u_{i+1} belong to the same groupoid $\mathcal{H}$ or $\mathcal{V}$.

These operations generate an equivalence relation in $\operatorname{Path}(\mathcal{Q})$. Following the lines of the proof of [H71, Theorem 5, Chapter 11], it is possible to see that in any equivalence class there is a unique reduced path.

The set of all reduced paths on $\mathcal{Q}$ forms a groupoid under the operation of concatenation followed by reduction. Compare also with the analogous construction for groups [S82, p. 186].

Also, the set of all reduced paths on $\mathcal{Q}$ with this product is isomorphic to $\mathcal{V} * \mathcal{H}$. Clearly $\mathcal{V} * \mathcal{H}$ contains both $\mathcal{V}$ and $\mathcal{H}$ as wide subgroupoids. In conclusion, any element u of $\mathcal{V} * \mathcal{H}$ has a unique standard form, namely:

- $u \in \mathcal{P}$ (elements of length 0), or
- $u = u_1 u_2 \dots u_n$, where the u_i's belong alternatively to different groupoids $\mathcal{V}$ or $\mathcal{H}$, no u_i is an identity (elements of length $n > 0$).

In such case we shall say that u_1, respectively u_n, is the *first*, respectively the *last*, letter of u.

Lemma 2.5. *Let $p = p_1 \dots p_N$, $q = q_1 \dots q_M$ be reduced paths in $\mathcal{V} * \mathcal{H}$ of lengths N and M respectively.*

(i). If p_N and q_1 belong to different groupoids $\mathcal{V}$ or $\mathcal{H}$, then length $(pq) = N + M$.

(ii). If p_N and q_1 belong to the same groupoid $\mathcal{V}$ or $\mathcal{H}$, but $p_N \neq (q_1)^{-1}$ then length $(pq) = N + M - 1$.

(iii). If p_N and q_1 belong to the same groupoid $\mathcal{V}$ or $\mathcal{H}$, $p_N = (q_1)^{-1}$ but $p_{N-1} \neq (q_2)^{-1}$ then length $(pq) = N + M - 2$. $\square$

2.3. **Diagonal groupoid of a slim double groupoid.**
Let $\mathcal{B}$ be a double groupoid. In the free product $\mathcal{V} * \mathcal{H}$, we denote

$$[A] := xgy^{-1}h^{-1}, \quad \text{if} \quad A = h \overset{x}{\underset{y}{\square}} g \in \mathcal{B}.$$

Define the 'diagonal' groupoid $\mathcal{D}(\mathcal{B})$ to be the quotient of the free product $\mathcal{V} * \mathcal{H}$ modulo the relations $[A]$, $A \in \mathcal{B}$.

In the rest of this section we suppose that $\mathcal{B}$ is slim and satisfies the filling condition 0.1.

Lemma 2.6. *The subgroupoid J generated by all relations $[A]$, $A \in \mathcal{B}$, is a normal subgroup bundle of the free product $\mathcal{V} * \mathcal{H}$.*

Hence, if $\mathcal{B}$ satisfies 0.1, then $\mathcal{D}(\mathcal{B}) = (\mathcal{V} * \mathcal{H})/J$.

Proof. It is clear that J is a subgroup bundle of $\mathcal{V} * \mathcal{H}$. Let $A = h \overset{x}{\underset{y}{\square}} g$ be a box in $\mathcal{B}$. We shall show that the expressions $z(xgy^{-1}h^{-1})z^{-1}$ and $f(xgy^{-1}h^{-1})f^{-1}$ both belong to J, for all $z \in \mathcal{H}$, $f \in \mathcal{V}$, such that $r(z) = l(x) = b(f)$. This implies normality because $\mathcal{V}$ and $\mathcal{H}$ generate $\mathcal{V} * \mathcal{H}$. Let z, f as above. We have $r(z) = l(x) = t(h)$. Hence, since $\mathcal{B}$ satisfies 0.1, we

may pick a box $r\,\underset{s}{\overset{z}{\square}}\,h$ in $\mathcal{B}$. Then the horizontal composition

$$r\,\underset{s}{\overset{z}{\square}}\,h\;h\,\underset{y}{\overset{x}{\square}}\,g = r\,\underset{sy}{\overset{zx}{\square}}\,g$$

is in $\mathcal{B}$. Therefore, the expressions $X = zhs^{-1}r^{-1}$ and $Y = zxgy^{-1}s^{-1}r^{-1}$ both belong to J. Hence so does the product $YX^{-1} = z(xgy^{-1}h^{-1})z^{-1}$. To show that $f(xgy^{-1}h^{-1})f^{-1}$ belongs to J, we argue as before, now picking a box $B = f\,\underset{x}{\overset{u}{\square}}\,v$ in $\mathcal{B}$ and then taking the vertical composition $\begin{matrix} B \\ A \end{matrix}$ in $\mathcal{B}$. $\square$

Composing the inclusions $\mathcal{H}, \mathcal{V} \to \mathcal{V} * \mathcal{H}$ with the canonical projection $\mathcal{V} * \mathcal{H} \to \mathcal{D}$ we get canonical groupoid maps $i : \mathcal{H} \to \mathcal{D}$, $j : \mathcal{V} \to \mathcal{D}$.

The diagram $(\mathcal{D}(\mathcal{B}), j, i)$ is characterized by the following universal property: for every diagram $(\mathcal{G}, j_0, i_0)$ over $\mathcal{V}$ and $\mathcal{H}$, such that $i_0(x)j_0(g) = j_0(h)i_0(y)$, whenever the box $h\,\underset{y}{\overset{x}{\square}}\,g$ is in $\mathcal{B}$, there is a unique morphism of diagrams $f : \mathcal{D}(\mathcal{B}) \to \mathcal{G}$.

It is clear that the assignment $\mathcal{B} \to (\mathcal{D}(\mathcal{B}), j, i)$ is functorial.

The filling condition on $\mathcal{B}$ corresponds to the factorizability condition on $\mathcal{D}(\mathcal{B})$, as we show next.

Lemma 2.7. $\mathcal{D}(\mathcal{B}) = j(\mathcal{V})i(\mathcal{H})$.

In particular, if $\mathcal{B}$ is finite then $\mathcal{D}(\mathcal{B})$ is a *finite* groupoid.

Proof. Every element in $\mathcal{V} * \mathcal{H}$ writes as a product $w_1 \dots w_m$, where each w_i is an element of $\mathcal{H}$ or $\mathcal{V}$. Hence every element in $\mathcal{D}$ factorizes as $\overline{w_1} \dots \overline{w_m}$, where $\overline{w_i}$ is the image of w_i under the canonical projection, which coincides with $i(w_i)$ or $j(w_i)$ according to whether w_i is an element of $\mathcal{H}$ or $\mathcal{V}$.

Therefore it is enough to see that every product $i(x)j(g)$, $x \in \mathcal{H}, g \in \mathcal{V}$, belongs to $j(\mathcal{V})i(\mathcal{H})$. Indeed, this implies that the elements in the factorization may be appropriately reordered to get an element in $j(\mathcal{V})i(\mathcal{H})$. To see this we use the assumption of condition 0.1 on $\mathcal{B}$: since the product $i(x)j(g)$,

$x \in \mathcal{H}, g \in \mathcal{V}$, is defined, then there is a box $\begin{matrix} & x & \\ h & \square & g \\ & y & \end{matrix}$ in $\mathcal{B}$. By construction of $\mathcal{D}$, $i(x)j(g) = j(h)i(y)$. The lemma follows. $\square$

2.4. **Main result.**

We can now prove the main result of this section. Lemma 2.9 encapsulates the most delicate part of the proof.

Theorem 2.8. *The assignments $\mathcal{B} \mapsto \mathcal{D}(\mathcal{B})$ and $\mathcal{D} \mapsto \square(\mathcal{D}, j, i)$ determine mutual category equivalences between*

(a) *The category of slim double groupoids* $\begin{matrix} \mathcal{B} & \rightrightarrows & \mathcal{H} \\ \downdownarrows & & \downdownarrows \\ \mathcal{V} & \rightrightarrows & \mathcal{P} \end{matrix}$ *satisfying the filling condition 0.1, with fixed $\mathcal{V}$ and $\mathcal{H}$, and*

(b) *The category of $(\mathcal{V}, \mathcal{H})$-factorizations of groupoids $\mathcal{D}$ on $\mathcal{P}$.*

Proof. It remains to show that the assignments are mutually inverse. Suppose first that $\mathcal{D} = j(\mathcal{V})i(\mathcal{H})$ is a factorization as in (b). Let $\mathcal{D}'$ be the diagonal groupoid associated to the double groupoid $\square(\mathcal{D}, j, i)$; so that there are groupoid maps $j' : \mathcal{V} \to \mathcal{D}'$, $i' : \mathcal{H} \to \mathcal{D}'$ such that $\mathcal{D}' = j'(\mathcal{V})i'(\mathcal{H})$, in view of Lemma 2.7.

The universal property of $\mathcal{D}'$ implies the existence of a unique groupoid map $f : \mathcal{D}' \to \mathcal{D}$ such that $fi' = i$ and $fj' = j$. Moreover, f is surjective because of the condition $\mathcal{D} = j(\mathcal{V})i(\mathcal{H})$. To prove injectivity of f, let $P \in \mathcal{P}$ and let $z \in \mathcal{D}'(P)$ such that $f(z) \in \mathcal{P}$. Write $z = j'(g)i'(x)$, $g \in \mathcal{V}$, $x \in \mathcal{H}$, such that $b(g) = l(x)$. Applying f to this identity, we get $\mathrm{id}_P = f(z) = j(g)i(x)$. In particular $t(g) = P = r(x)$, and $i(\mathrm{id}_P)j(\mathrm{id}_P) = j(g)i(x)$ in $\mathcal{D}$.

The definition of $\square(\mathcal{D}, j, i)$ implies that the box $\begin{matrix} & = & \\ g & \square & \| \\ & x & \end{matrix}$ is in $\square(\mathcal{D}, j, i)$. Therefore, in view of the defining relations in $\mathcal{D}'$, we have $z = j'(g)i'(x) = \mathrm{id}_P$. This proves that f is injective and thus an isomorphism.

Let now $\mathcal{B}$ be a double groupoid as in (a), $\mathcal{D} = \mathcal{D}(\mathcal{B})$ the associated diagonal groupoid with the canonical maps $i : \mathcal{H} \to \mathcal{D}$, $j : \mathcal{V} \to \mathcal{D}$, and $\mathcal{B}' = \square(\mathcal{D}, j, i)$. Let $\begin{matrix} & x & \\ h & \square & g \\ & y & \end{matrix}$ be a box in $\mathcal{B}$. Then $i(x)j(g) = j(h)i(y)$ in

$\mathcal{D}$ and this relation determines a unique box $h \overset{x}{\underset{y}{\square}} g$ in $\mathcal{B}'$, since $\mathcal{B}'$ is slim. This defines a map $F : \mathcal{B} \to \mathcal{B}'$ that, because of the slim condition on $\mathcal{B}$, turns out to be an injective map of double groupoids.

We claim that F is also surjective, hence an isomorphism. To establish this claim, we shall need the presentation of the free product $\mathcal{V} * \mathcal{H}$ given in Subsection 2.2. Let $A = h \overset{x}{\underset{y}{\square}} g$ be a box in $\mathcal{B}'$, which means that

$$i(x)j(g) = j(h)i(y) \tag{2.1}$$

in $\mathcal{D}$. We shall prove that there is a box $h \overset{x}{\underset{y}{\square}} g$ in $\mathcal{B}$. First, using the filling condition 0.1 in $\mathcal{B}$, there is a box $A_0 = h_0 \overset{x}{\underset{y_0}{\square}} g \in \mathcal{B}$. Then it is enough to show that the box $E = {}^{f}\underset{z}{\square}$ is also in $\mathcal{B}$, where $f = h_0^{-1}h$ and $z = yy_0^{-1}$. In fact, if $E \in \mathcal{B}$, then $E \rightharpoondown A_0 \in \mathcal{B}$ is the desired box.

Since A_0 belongs to $\mathcal{B}$, $i(x)j(g) = j(h_0)i(y_0)$ in $\mathcal{D}$; combined with (2.1), this gives $j(f)i(z) \in \mathcal{P}$. The definition of $\mathcal{D}$ combined with Lemma 2.6 implies that the path fz belongs to the normal group bundle J. The proof of the Theorem will be achieved once the following Lemma is established. $\square$

Lemma 2.9. *Let $f \in \mathcal{V}$ and $z \in \mathcal{H}$ such that*

- $t(f) = r(z) =: P$ *and* $b(f) = l(z)$.
- *There exist* $A_1, \dots, A_n \in \mathcal{B}$, $\epsilon_1, \dots, \epsilon_n \in \{\pm 1\}$ *such that*

$$fz = [A_1]^{\epsilon_1} \dots [A_n]^{\epsilon_n}. \tag{2.2}$$

Then there exists $E \in \mathbf{E}$ such that $E = {}^{f}\underset{z}{\square}$.

Proof. We proceed by induction on n. Note that the case $n = 0$ means that $fz = \mathrm{id}_P$ in $\mathcal{V} * \mathcal{H}$, so that $f = \mathrm{id}_P^{\mathcal{V}}$, $z = \mathrm{id}_P^{\mathcal{H}}$, since the word fz is not reduced; thus $E = \Theta_P$ is the desired element in $\mathcal{P}$.

Proof when $n = 1$. We need to work out this case by technical reasons, see Sublemma 2.11 below. We shall assume that $f \notin \mathcal{P}$, $z \notin \mathcal{P}$, so that the

path fz is reduced; if either $f \in \mathcal{P}$ or $z \in \mathcal{P}$ the arguments are similar. We have $fz = [A_1]^{\epsilon_1}$. If $\epsilon_1 = 1$ this says that $fz = xgy^{-1}h^{-1}$ where we omit the subscript 1 for simplicity. Hence, the right-hand side is not reduced. Since the letters x, g, y, h belong alternatively to different groupoids, at least one of them should be in $\mathcal{P}$. If $x = \mathrm{id}_{\mathcal{H}} P$, then $fz = gy^{-1}h^{-1}$ hence necessarily $h = \mathrm{id}_{\mathcal{V}} P$, $f = g$, $z = y^{-1}$ and $A_1 = \underset{z^{-1}}{\boxed{}}\,f$. Thus $E = (A_1)^h$ does the job. If $x \notin \mathcal{P}$ then it should be cancelled because the left-hand side begins by $f \in \mathcal{V}$; thus $g \in \mathcal{P}$, $y = x^{-1}$ and $fz = h$, a contradiction.

Assume now that $\epsilon_1 = -1$, that is, $fz = hyg^{-1}x^{-1}$. Again, the right-hand side is not reduced and one of the letters should be in $\mathcal{P}$. If $h = \mathrm{id}_{\mathcal{V}} P$ then $fz = yg^{-1}x^{-1}$ hence necessarily $y = \mathrm{id}_{\mathcal{H}} P$, $f = g^{-1}$, $z = x^{-1}$ and $E = (A_1)^{-1}$ does the job. If $h \notin \mathcal{P}$ then either

(a) $y \in \mathcal{P}$, $f = hg^{-1}$, $z = x^{-1}$ and $E = \begin{Bmatrix} \mathrm{id}\, h \\ (A_1)^{-1} \end{Bmatrix}$ does the job; or

(b) $y \notin \mathcal{P}$, $g \in \mathcal{P}$, $f = h$, $z = yx^{-1}$ and $E = A_1 \ \mathrm{id}\, x^{-1}$ does the job.

Assume now that the claim is true for $n - 1$. Assume that (2.2) holds for n. Our aim is to reduce the right-hand side of (2.2) to $n - 1$ factors, or to achieve a contradiction by comparison of the lengths in both sides.

Before we begin to analyze contiguous brackets, where 'bracket' means an element of the form $[A_i]^{\pm 1}$, let us set up some preliminaries. Let us say that $A_i \in \mathcal{B}$ is of *class* ℓ if exactly ℓ of its sides are not in $\mathcal{P}$. We shall assign a 'type' to any bracket $[A_i]^{\pm 1}$.

- If $\ell = 0$, all the sides of A_i are in $\mathcal{P}$; hence $[A_i]^{\epsilon_i}$ can be extirpated from the right-hand side of (2.2) and we are done by the inductive hypothesis. Hence, we can assume that $\ell > 0$ for all i.

- If $\ell = 1$, we distinguish two types.

(α): A_i has a non-trivial vertical side. We can assume that $A_i = h_i\boxed{}$.

For, if $A_i = \boxed{}\,g_i$ then $[A_i] = [\widetilde{A}_i]$ where $\widetilde{A}_i = \begin{Bmatrix} \mathrm{id}\, g_i^{-1} \\ A_i \end{Bmatrix} = g_i^{-1}\boxed{}$.

Moreover if $[A_i]$ is of type (α) then $[A_i]^{-1} = [A_i^v]$, again of type (α).

(β): A_i has a non-trivial horizontal side. We can assume that $A_i = \overset{x_i}{\boxed{}}$.

Moreover if $[A_i]$ is of type (β) then $[A_i]^{-1} = [A_i^h]$, again of type (β). In particular, if $\ell = 1$, then we can assume $\epsilon_i = 1$.

- If $\ell = 2$, we can assume that the two non-trivial sides live in different groupoids. For, if $A_i = h_i \square\, g_i$ then $[A_i] = [\widetilde{A}_i]$ where $\widetilde{A}_i = \begin{Bmatrix} \operatorname{id} g_i^{-1} \\ A_i \end{Bmatrix} = g_i^{-1} h_i \square$, whose bracket is of type (α). Similarly, if A_i has two non-trivial horizontal sides then $[A_i]^{\epsilon_i}$ can be replaced by a bracket of type (β).

We distinguish two types.

(γ): $[A_i] \in \mathcal{HV}$. We can assume that $A_i = h_i \overset{x_i}{\square}$. For, if $A_i = \overset{x_i}{\square}\, g_i$ then

$[A_i] = [\widetilde{A}_i]$ where $\widetilde{A}_i = \begin{Bmatrix} A_i \\ \operatorname{id} g_i^{-1} \end{Bmatrix} = g_i^{-1} \overset{x_i}{\square}$. Similarly, if $A_i = h_i \underset{y_i}{\square}$ then

$[A_i] = [\widetilde{A}_i]$ where $\widetilde{A}_i = A_i \operatorname{id} y_i^{-1} = \quad h_i \overset{y_i^{-1}}{\square} \quad .$

(δ): $[A_i] \in \mathcal{VH}$. Then $A_i = \underset{y_i}{\square}\, g_i$.

Moreover if $[A_i]$ is of type (γ) then $[A_i]^{-1} = [B]$, where $B = (A_i \operatorname{id} t(A_i)^{-1})^h$ is of type (δ). In particular, if $\ell = 2$ then we can assume $\epsilon_i = 1$.

- If $\ell = 3$, we can assume that the identity sides are either x_i or h_i; otherwise we replace the box by one with $\ell = 2$. There are two types.

$$(\zeta): \quad A_i = \overset{x_i}{\underset{y_i}{\square}}\, g_i, \qquad (\eta): \quad A_i = h_i \underset{y_i}{\square}\, g_i .$$

Moreover if $[A_i]$ is of type (ζ) then $[A_i]^{-1} = [A_i^v]$ is again of type (ζ); if $[A_i]$ is of type (η) then $[A_i]^{-1} = [A_i^h]$ is again of type (η). In particular, if $\ell = 3$ then we can assume $\epsilon_i = 1$.

- If $\ell = 4$, we distinguish two types.

$$(\theta): \quad \epsilon = 1, \qquad (\kappa): \quad \epsilon = -1.$$

In conclusion the right-hand side of (2.2) *is a product of n brackets of types* (α), ..., (κ) *with the exponent* $\epsilon_i = 1$ *except for the type* (κ).

Next, we can assume several restrictions on the contiguity of these brackets, as summarized in the following statement. We prove below that, whenever these restrictions do not hold, then we can replace the pair of contiguous brackets by a single bracket and hence apply the inductive hypothesis.

Restrictions 2.10. *Let* $1 \leq i \leq n$.

(1). *If* $[A_i]$ *is of type* (α), *then* $i > 1$ *and* $[A_{i-1}]^{\epsilon_{i-1}}$ *is of type* (κ). *Also, if* $i < n$ *then* $[A_{i+1}]^{\epsilon_{i+1}}$ *is of type* (θ).

(2). *If* $[A_i]$ *is of type* (β), *then* $i < n$ *and* $[A_{i+1}]^{\epsilon_{i+1}}$ *is of type* (κ). *Also, if* $i > 1$ *then* $[A_{i-1}]^{\epsilon_{i-1}}$ *is of type* (θ).

(3). *If* $[A_i]$ *is of type* (γ), *then:*

- **(a):** *If* $i > 1$ *then* $[A_{i-1}]^{\epsilon_{i-1}}$ *is of type* (η), (θ), *or* (κ) *with* $x_{i-1} \neq x_i$.
- **(b):** *If* $i < n$ *then* $[A_{i+1}]^{\epsilon_{i+1}}$ *is of type* (ζ), (θ), *or* (κ) *with* $h_{i+1} \neq h_i$.

(4). *If* $[A_i]$ *is of type* (δ), *then:*

- **(a):** *If* $i > 1$ *then* $[A_{i-1}]^{\epsilon_{i-1}}$ *is of type* (ζ), (θ), (κ), *or* (θ) *with* $h_{i-1} \neq g_i$.
- **(b):** *If* $i < n$ *then* $[A_{i+1}]^{\epsilon_{i+1}}$ *is of type* (η), (κ), *or* (θ) *with* $x_{i+1} \neq y_i$.

(5). *If* $[A_i]$ *is of type* (ζ), *then:*

- **(a):** *If* $i > 1$ *then* $[A_{i-1}]^{\epsilon_{i-1}}$ *is of type* (γ), (ζ), (η), (θ) *or* (κ). *Also,*
  (1) *if* $[A_{i-1}]$ *is of type* (ζ) *then* $y_{i-1} \neq x_i$;
  (2) *if* $[A_{i-1}]^{\epsilon_{i-1}}$ *is of type* (κ) *then* $x_{i-1} \neq x_i$.
- **(b):** *If* $i < n$ *then* $[A_{i+1}]^{\epsilon_{i+1}}$ *is of type* (δ), (ζ), (η), (θ) *or* (κ). *Also,*
  (1) *if* $[A_{i+1}]$ *is of type* (ζ) *then* $y_i \neq x_{i+1}$;
  (2) *if* $[A_{i+1}]^{\epsilon_{i+1}}$ *is of type* (θ) *then* $y_i \neq x_{i+1}$.

(6). *If* $[A_i]$ *is of type* (η), *then:*

- **(a):** *If* $i > 1$ *then* $[A_{i-1}]^{\epsilon_{i-1}}$ *is of type* (δ), (ζ), (η), (θ) *or* (κ). *Also,*
  (1) *if* $[A_{i-1}]$ *is of type* (η) *then* $h_{i-1} \neq g_i$;
  (2) *if* $[A_{i-1}]^{\epsilon_{i-1}}$ *is of type* (θ) *then* $h_{i-1} \neq g_i$.
- **(b):** *If* $i < n$ *then* $[A_{i+1}]^{\epsilon_{i+1}}$ *is of type* (γ), (ζ), (η), (θ) *or* (κ). *Also,*
  (1) *if* $[A_{i+1}]$ *is of type* (η) *then* $g_{i+1} \neq h_i$;
  (2) *if* $[A_{i+1}]^{\epsilon_{i+1}}$ *is of type* (κ) *then* $h_{i+1} \neq h_i$.

(7). *If* $[A_i]^{\epsilon_i}$ *and* $[A_{i+1}]^{\epsilon_{i+1}}$ *are both of types* (θ) *or* (κ) *for some* i, $1 \leq i \leq n-1$, *then either*

- $\epsilon_i = \epsilon_{i+1}$; *or*
- $\epsilon_i = 1 = -\epsilon_{i+1}$, *and* $h_i \neq h_{i+1}$; *or*
- $\epsilon_i = 1 = -\epsilon_{i+1}$, $h_i = h_{i+1}$ *and* $y_i \neq y_{i+1}$; *or*
- $\epsilon_i = -1 = -\epsilon_{i+1}$ *and* $x_i \neq x_{i+1}$; *or*
- $\epsilon_i = -1 = -\epsilon_{i+1}$, $x_i = x_{i+1}$ *and* $g_i \neq g_{i+1}$.

We deal with (1). If $[A_1]$ is of type (α) then $h_1 f z = [A_2]^{\epsilon_2} \dots [A_n]^{\epsilon_n}$; by the inductive hypothesis there exists $E_1 \in \mathbf{E}$ such that $E_1 = \begin{array}{cc} h_1 f & \square \\ & z \end{array}$. Then $E = \begin{Bmatrix} (A_1)^v \\ E_1 \end{Bmatrix}$ does the job. Thus we do not consider this possibility. Assume that $i > 1$ and $[A_{i-1}]^{\epsilon_{i-1}}$ is not of type (κ). We show that $[A_{i-1}][A_i] = [B]$; hence the right-hand side of (2.2) has really $n-1$ factors and the existence of E follows from the inductive hypothesis. Thus we do not consider this possibility. Explicitly, $B \in \mathcal{B}$ is given as follows.

- If $[A_{i-1}]$ is of type (α) or (η) then $B = \begin{Bmatrix} A_i \\ A_{i-1} \end{Bmatrix}$.
- If $[A_{i-1}]$ is of type (β) then $B = \begin{Bmatrix} A_{i-1} \\ A_i \end{Bmatrix}$.
- If $[A_{i-1}]$ is of type (γ) then $B = \begin{Bmatrix} A_i \\ \operatorname{id} h_{i-1}^{-1} \\ A_{i-1} \end{Bmatrix}$.
- If $[A_{i-1}]$ is of type (δ) or (ζ) then $B = A_i\, A_{i-1}$.
- If $[A_{i-1}]$ is of type (θ) then $B = \begin{Bmatrix} A_i \\ A_{i-1}C \end{Bmatrix} C^h$, where $C \in \mathcal{B}$ is a box filling the configuration $\begin{array}{c|c} & x_i^{-1} \\ \hline g_i & \end{array}$.

Now, if $i < n$ then write $[A_{i+1}]^{\epsilon_{i+1}} = [\widetilde{A}_{i+1}]^{-\epsilon_{i+1}}$. Then $[A_i][A_{i+1}]^{\epsilon_{i+1}} = \left([\widetilde{A}_{i+1}]^{-\epsilon_{i+1}}[A_i^v]\right)^{-1}$. We discard this possibility by the previous discussion unless $[\widetilde{A}_{i+1}]^{-\epsilon_{i+1}}$ is of type (κ), that is, unless $[A_{i+1}]^{\epsilon_{i+1}}$ is of type (θ).

We deal with (2). If $[A_n]$ is of type (β) then $f z x_n^{-1} = [A_1]^{\epsilon_1} \dots [A_{n-1}]^{\epsilon_{n-1}}$; by the inductive hypothesis there exists $E_1 \in \mathbf{E}$ such that $E_1 = \begin{array}{cc} f & \square \\ & z x_n^{-1} \end{array}$.

Then $E = E_1(A_n)^v$ does the job. Thus we do not consider this possibility. Assume that $i < n$ and $[A_{i+1}]^{\epsilon_{i+1}}$ is not of type (κ). We show that $[A_i][A_{i+1}] = [B]$; hence the right-hand side of (2.2) has really $n-1$ factors and the existence of E follows from the inductive hypothesis. Thus we do not consider this possibility. Explicitly, $B \in \mathcal{B}$ is given as follows.

- If $[A_{i+1}]$ is of type (α) or (η) then $B = \left\{\begin{matrix} A_i \\ A_{i+1} \end{matrix}\right\}$.
- If $[A_{i+1}]$ is of type (β) or (δ) or (ζ) then $B = A_i\, A_{i+1}$.
- If $[A_{i+1}]$ is of type (γ) then $B = A_i \left\{\begin{matrix} A_{i+1} \\ \operatorname{id} h_{i+1}^{-1} \end{matrix}\right\}$.
- If $[A_{i+1}]$ is of type (θ) then $B = \left\{\begin{matrix} A_i \left\{\begin{matrix} A_{i+1} \\ C \end{matrix}\right\} \\ C^v \end{matrix}\right\}$, where $C \in \mathcal{B}$ is a box filling the configuration $\begin{array}{c|c} & y_{i+1} \\ \hline h_{i+1}^{-1} & \end{array}$.

If $i > 1$, we may write $[A_{i-1}]^{\epsilon_{i-1}} = [\widetilde{A}_{i-1}]^{-\epsilon_{i-1}}$, hence $[A_{i-1}]^{\epsilon_{i-1}}[A_i] = \Big([A_i^h][\widetilde{A}_{i-1}]^{-\epsilon_{i-1}}\Big)^{-1}$. We discard this, unless $[A_{i-1}]^{\epsilon_{i-1}}$ is of type (θ).

We deal with (3a). We may assume that $[A_{i-1}]^{\epsilon_{i-1}}$ is neither of type (α) nor (β) by (1) and (2). If $[A_{i-1}]^{\epsilon_{i-1}}$ is of type (γ), (δ), (ζ), or (κ) with $x_{i-1} = x_i$, then $[A_{i-1}]^{\epsilon_{i-1}}[A_i] = [B]$; hence the right-hand side of (2.2) has $n-1$ factors and, by the inductive hypothesis, we do not consider this possibility. Here $B \in \mathcal{B}$ is:

- If $[A_{i-1}]$ is of type (γ) then $B = \left\{\begin{matrix} A_{i-1} \\ \operatorname{id} h_{i-1}^{-1} \\ \left\{A_i \operatorname{id} x_i^{-1}\right\} \end{matrix}\right\}$.
- If $[A_{i-1}]$ is of type (δ) or (ζ) then $B = A_i\, \operatorname{id} x_i^{-1}\, A_{i-1}$.
- If $[A_{i-1}]^{-1}$ is of type (κ) with $x_{i-1} = x_i$, then $B = \left\{\begin{matrix} A_i \operatorname{id} x_1^{-1} \\ A_{i-1} \end{matrix}\right\}$.

We deal with (3b). We may assume that $[A_{i+1}]^{\epsilon_{i+1}}$ is neither of type (α) nor (β) nor (γ) by (1), (2) and (3a). If $[A_{i+1}]^{\epsilon_{i+1}}$ is of type (δ), (η), or (κ) with $h_i = h_{i+1}$ then $[A_i][A_{i+1}]^{\epsilon_{i+1}} = [B]$. By the inductive hypothesis, we do not consider this possibility. Here $B \in \mathcal{B}$ is:

- If $[A_{i+1}]$ is of type (δ) or (η) then $B = \left\{\begin{matrix} A_i \\ \operatorname{id} h_i^{-1} \\ A_{i+1} \end{matrix}\right\}$.

- If $[A_{i+1}]^{-1}$ is of type (κ) with $h_i = h_{i+1}$ then $B = \begin{Bmatrix} A_i \\ \mathrm{id}\, h_i^{-1} \end{Bmatrix} A^v_{i+1}$.

Now (4a) follows from (3b), and (4b) follows from (3a), by the inversion argument as for the second parts of (1) and (2).

We deal with (5a). If $[A_{i-1}]$ is of type (ζ) and $y_{i-1} = x_i$, then $[A_{i-1}][A_i] = [B]$ where $B = \begin{Bmatrix} A_{i-1} \\ A_i \end{Bmatrix}$. If $[A_{i-1}]^{-1}$ is of type (κ) and $x_{i-1} = x_i$, then $[A_{i-1}]^{-1}[A_i] = [B]^{-1}$ where $B = \begin{Bmatrix} A^v_i \\ A_{i-1} \end{Bmatrix}$. By the inductive hypothesis, we do not consider this possibility. Now (5b) follows from (5a) by the inversion argument.

We deal with (6a). If $[A_{i-1}]$ is of type (η) and $h_{i-1} = g_i$, then $[A_{i-1}][A_i] = [B]$ where $B = A_i A_{i-1}$. If $[A_{i-1}]$ is of type (θ) and $h_{i-1} = g_i$, then $[A_{i-1}][A_i] = [B]$ where $B = A_i A_{i-1}$. By the inductive hypothesis, we do not consider this possibility. Now (6b) follows from (6a) by the inversion argument.

We deal with (7). If $\epsilon_i = 1 = -\epsilon_{i+1}$, $h_i = h_{i+1}$ and $y_i = y_{i+1}$, then $[A_i][A_{i+1}]^{-1} = [B]$ where $B = \begin{Bmatrix} A_i \\ A^v_{i+1} \end{Bmatrix}$.

If $\epsilon_i = -1 = -\epsilon_{i+1}$, $x_i = x_{i+1}$ and $g_i = g_{i+1}$, then $[A_i]^{-1}[A_{i+1}] = [B]^{-1}$ where $B = A_i A^h_{i+1}$.

To conclude the proof of the lemma, and *a fortiori* of the theorem, we need to establish the following fact.

Sublemma 2.11. *Let $n \geq 2$. Consider an element $P_n \in \mathcal{V} * \mathcal{H}$, such that*

$$P_n = [A_1]^{\epsilon_1} \dots [A_n]^{\epsilon_n}, \tag{2.3}$$

where $A_1, \dots, A_n \in \mathcal{B}$, $\epsilon_1, \dots, \epsilon_n \in \{\pm 1\}$; the brackets $[A_i]^{\epsilon_i}$ are of type (α), …, (κ), $1 \leq i \leq n$; and contiguous brackets satisfy Restrictions 2.10. Then

(i) the last letter of P_n is the last letter of $[A_n]^{\epsilon_n}$,

(ii) if $[A_n]^{\epsilon_n}$ has type (θ) or (κ) then the last two letters of P_n equal the last two of $[A_n]^{\epsilon_n}$, and

(iii) the length of $P_n > 2$.

Proof. If $n = 2$ the claim follows by inspection of the possible cases; in fact, the length of P_2 turns out to be ≥ 5. Now assume that the claim is true

for $n \geq 2$. Consider P_{n+1} as in (2.3) and write $P_{n+1} = P_n[A_{n+1}]^{\epsilon_{n+1}}$. By hypothesis, the length of P_2 is > 2 and its last letter is that of $[A_n]^{\epsilon_n}$. Now the pair $[A_n]^{\epsilon_n}[A_{n+1}]^{\epsilon_{n+1}}$ satisfies Restrictions 2.10. The sublemma now follows from Lemma 2.5. □

We can now finish the proof of the Lemma. If the right-hand side of (2.2) satisfies Restrictions 2.10, then the sublemma gives a contradiction, since the left-hand side has length ≤ 2. Thus at least one bracket can be eliminated in the right-hand side and the Lemma follows by induction. □

3. Fusion double groupoids

Let $\mathcal{B}$ be a finite double groupoid satisfying the filling condition 0.1. The following definition is motivated by [AN06, Proposition 3.16].

Definition 3.1. We say that $\mathcal{B}$ is *fusion* if and only if the following hold:

(F1) The vertical groupoid $\mathcal{V} \rightrightarrows \mathcal{P}$ is connected.

(F2) For any $x \in \mathcal{H}$, there exists at most one $E \in \mathbf{E}$ such that $b(E) = x$.

Observe that condition (F2) in the definition is equivalent to injectivity of the morphism $b : \mathbf{E} \to \mathcal{H}$ of groupoids over $\mathcal{P}$.

Let $\mathbf{E}$ be the core groupoid of $\mathcal{B}$ and let $\mathfrak{E}$ be the core groupoid of its frame $\mathcal{F}$. Then there is an exact sequence of groupoids

$$1 \longrightarrow \mathbf{K} \longrightarrow \mathbf{E} \longrightarrow \mathfrak{E} \longrightarrow 1.$$

In particular, for any $P \in \mathcal{P}$, we have

$$|\mathbf{E}(P)| = |\mathbf{K}(P)||\mathfrak{E}(P)|. \tag{3.1}$$

Proposition 3.2. *Suppose that $\mathcal{B}$ is fusion. Then $\mathcal{B}$ is slim.*

Proof. It follows immediately from condition (F2) in Definition 3.1. □

Then, in view of Theorem 2.8, a fusion double groupoid $\mathcal{B}$ is determined by a $(\mathcal{V}, \mathcal{H})$-factorization of its diagonal groupoid $\mathcal{D} = \mathcal{D}(\mathcal{B})$.

Proposition 3.3. *The following are equivalent:*

(i) *$\mathcal{B}$ is fusion.*
(ii) *$\mathcal{V} \rightrightarrows \mathcal{P}$ is connected and $j : \mathcal{V} \to \mathcal{D}(\mathcal{B})$ is injective.*

In particular, if $\mathcal{B}$ is fusion, then its diagonal groupoid $\mathcal{D} = \mathcal{D}(\mathcal{B})$ is connected.

Proof. It is enough to see that injectivity of the map $b : \mathbf{E} \to \mathcal{H}$ is equivalent to injectivity of the map j. Suppose first that $b : \mathbf{E} \to \mathcal{H}$ is injective. Let $g \in \mathcal{V}$ such that $j(g) \in \mathcal{P}$. By Theorem 2.8, $\mathcal{B} \simeq \mathcal{B}(\mathcal{D}, j, i)$. Then there is a box $g\boxed{} \in \mathcal{B}$. Since this box belongs to the core groupoid $\mathbf{E}$, the injectivity of b implies that $g \in \mathcal{P}$. Hence j is injective.

Conversely, suppose that j is injective. Let $E \in \mathbf{E}$ such that $b(E) \in \mathcal{P}$, so that $E = g\boxed{}$, for some $g \in \mathcal{V}$. The construction of $\mathcal{D}$ implies that $j(g) \in \mathcal{P}$ and therefore $g \in \mathcal{P}$. The slim condition on $\mathcal{B}$ now guarantees that $E \in \mathcal{P}$. Thus b is injective. $\square$

Appendix A. Corner functions

A.1. **Orbits of groupoid actions.** We first discuss an extension of the basic counting formula for orbits in group theory to the setting of groupoids. Let $s, e : \mathcal{G} \rightrightarrows \mathcal{P}$ be a groupoid. Recall the equivalence relation induced by $\mathcal{G}$ on $\mathcal{P}$: $P \sim Q$ if and only if $\mathcal{G}(P, Q) \neq \emptyset$. We denote by $\widetilde{Q}$ the equivalence class of Q. We set

$$\mathcal{G}_{\curvearrowright Q} = \{g \in \mathcal{G} : e(g) = Q\} = \coprod_{P \in \mathcal{P}} \mathcal{G}(P, Q) = \coprod_{P \in \widetilde{Q}} \mathcal{G}(P, Q),$$

the set of arrows with target Q. It is clear from the above that

$$|\mathcal{G}_{\curvearrowright Q}| = |\widetilde{Q}| \times |\mathcal{G}(Q)|.$$

We shall consider the fiber bundle $s : \mathcal{G}_{\curvearrowright Q} \to \mathcal{P}$.

Let K be a subgroup of $\mathcal{G}(Q)$. We define the quotient $\mathcal{G}_{\curvearrowright Q}/K := \mathcal{G}_{\curvearrowright Q}/\equiv_K$, where $\equiv_K$ is the equivalence relation in $\mathcal{G}_{\curvearrowright Q}$ given by

$$g \equiv_K h \iff g^{-1}h \in K.$$

Clearly, the source map descends to the quotient and we can consider the fiber bundle $s : \mathcal{G}_{\curvearrowright Q}/K \to \mathcal{P}$. Hence, if $\mathcal{G}$ is finite, then

$$|\mathcal{G}_{\curvearrowright Q}/K| = \frac{|\widetilde{Q}| \times |\mathcal{G}(Q)|}{|K|}.$$

Clearly, $\mathcal{G}$ acts on $s : \mathcal{G}_{\curvearrowright Q} \to \mathcal{P}$ by left multiplication.

Assume that $\mathcal{G}$ acts on $p : \mathcal{E} \to \mathcal{P}$. The groupoid $\mathcal{G}$ still acts on the orbit $\mathcal{O}_x$. Then there is an isomorphism of $\mathcal{G}$-fiber bundles $\varphi : \mathcal{G}_{\to p(x)}/\mathcal{G}^x \to \mathcal{O}_x$ induced by $g \mapsto g \triangleright x$. In particular, if $\mathcal{G}$ is finite, then

$$|\mathcal{O}_x| = \frac{|\widetilde{p(x)}| \times |\mathcal{G}(p(x))|}{|\mathcal{G}^x|}. \tag{A.1}$$

A.2. **Corner functions.** Let $\mathcal{B}$ be a finite double groupoid. We apply the counting argument above to give alternative proofs of some properties of the 'corner' functions defined in [AN06]. There are four corner functions but it is enough to consider one. Recall the sets $\mathcal{UR}(x,g)$ and $\mathcal{UR}(B)$ with prescribed upper and right sides, as defined in (1.2). The 'upper-right' corner functions are $\urcorner : \mathcal{H}\,{}_r\!\times_t \mathcal{V} \to \mathbb{N}\cup\{0\}$ and $\urcorner : \mathcal{B} \to \mathbb{N}$ defined by

$$\urcorner(x,g) = |\mathcal{UR}(x,g)| \quad \text{and} \quad \urcorner(B) = |\mathcal{UR}(B)|.$$

The other corner functions are defined similarly. In this parlance, the *filling condition* on $\mathcal{B}$ is just

$$\urcorner(x,g) > 0 \quad \text{for any } (x,g) \in \mathcal{H}\,{}_r\!\times_t \mathcal{V}. \tag{A.2}$$

We now interpret the corner functions in terms of orbits of an action of the core groupoid. Let $\gamma : \mathcal{B} \to \mathcal{P}$ be the 'left-bottom' vertex, $\gamma(B) = lb(B)$. Let $B \in \mathcal{B}$ and $Q = \gamma(B) = bl(B)$. Consider the relation $\sim$ on $\mathcal{P}$ induced by $\mathbf{E}$. Recall that $\theta(Q)$ is the common value

$$\lrcorner(\mathrm{id}_{\mathcal{V}}\, Q, \mathrm{id}_{\mathcal{H}}\, Q) = \urcorner(\mathrm{id}_{\mathcal{V}}\, Q, \mathrm{id}_{\mathcal{H}}\, Q) = \llcorner(\mathrm{id}_{\mathcal{V}}\, Q, \mathrm{id}_{\mathcal{H}}\, Q) = \ulcorner(\mathrm{id}_{\mathcal{V}}\, Q, \mathrm{id}_{\mathcal{H}}\, Q).$$

The proposition 1.1, together with formula (A.1), implies the following formula for the corner function:

$$\urcorner(B) = |\widetilde{Q}| \times |\mathbf{E}(Q)|. \tag{A.3}$$

Applied to $B = \Theta_Q$, the formula implies that

$$\theta(Q) = \urcorner(\Theta_Q) = |\widetilde{Q}| \times |\mathbf{E}(Q)| = \urcorner(B).$$

Hence $\theta(Q)$ is also given by (A.3). That is, the corner functions on a box depend only on the vertex 'opposite' to the corner of that box. Formula (A.3) provides easy alternative proofs of the following facts– see [AN06]:

(a) Let $P, Q \in \mathcal{P}$. If $P \sim Q$, then $\theta(P) = \theta(Q)$.

(b) Let $L, M, N \in \mathcal{B}$. Suppose that $\begin{array}{c|c} L & M \\ \hline N & \end{array}$. Then

$$\ulcorner(L) = \urcorner(M), \quad \llcorner(L) = \lrcorner(M), \quad \ulcorner(L) = \llcorner(N), \quad \urcorner(L) = \lrcorner(N).$$

(c) Let $X, Y, Z \in \mathcal{B}$ such that $\begin{array}{c|c} X & Y \\ \hline Z & \end{array}$. Then

$$\urcorner(XY) = \urcorner(X), \quad \urcorner\begin{pmatrix} X \\ Z \end{pmatrix} = \urcorner(Z), \quad \llcorner(XY) = \llcorner(Y), \quad \llcorner\begin{pmatrix} X \\ Z \end{pmatrix} = \llcorner(X).$$

(d) The double groupoid is vacant if and only if the core groupoid is trivial.

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